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Mean-field theory and fluctuations in Potts spin glasses: I

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Abstract. In this and a subsequent paper a short-range Potts spin glass is studied. A stable, non-marginal, mean-field theory is found with one level of replica symmetry breaking and a discontinuous transition for $p > 4$. A complete stability analysis is provided, and two different correlation lengths are found above eight dimensions. The fluctuations in the ordered phase around this solution are incorporated in a renormalisation group approach and it is found that for a small range of the parameters they restore scaling close to the upper critical dimension. For the three-state case it appears that fluctuations destroy the stability of the solution for $d \leqslant 8$ and cause the system to undergo a first-order phase transition. Non-universal corrections to the equation of state above the upper critical dimension are discussed.

1. Introduction

Considerable attention has been devoted in recent years to Potts spin glasses [1-6]. There are several factors that make these systems interesting.

(i) Certain experimental systems like mixed ortho-para hydrogen crystals [7], electric dipole glasses [8] and orientational glasses like K(Br, Cn) mixed crystals [9] do not have the reflection symmetry of an Ising system. A random Potts system seems a more reasonable starting point to model them.

(ii) Although the model possesses both randomness and frustration, which are now believed to be necessary ingredients of spin glass behaviour (for a review of spin glasses, see [10]), the degree of frustration of Potts random systems is smaller than their Ising counterpart [11]; for example, the weight of the frustrated loop configurations decays exponentially with the length of the loop, while in the Ising spin glasses (ISG), frustration is present at all lengths. It is reasonable to think that some of the simplifying features of the mean-field solution described below are connected with this fact.

(iii) For $p < 4$ (p being the number of equivalent Potts states) the mean-field solution obtained shows a continuous transition to a spin-glass phase which for a certain range of temperature is stable, in the sense that all the eigenvalues of fluctuations about it are strictly positive. Although the replica symmetric solution is unstable in the Potts glass, as in the case of the ISG [12], the stable mean-field solution requires a much simpler ansatz of replica symmetry breaking than its counterpart in the ISG [13], and does not have the high degree of marginality that the latter presents. Because of this fact, the effect of the fluctuations in the ordered phase can be explicitly incorporated.

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(iv) For $p>4$ the mean-field theory indicates a transition to a spin-glass phase with a discontinuous jump of the Edwards–Anderson order parameter [14] at T_g. Although at the transition all the perturbative fluctuations remain finite this is not a regular first-order transition. Only the second derivative of the free energy is discontinuous at T_g; in particular there is no latent heat at the transition. Using an extension of the Thouless–Anderson–Palmer [15] (TAP) approach to this model, it has been argued [5] that this is related to the fact that the number of states into which the system can freeze is exponentially large at T_g, and, thus it does not become more ordered in a statistical sense when it freezes.

(v) It is interesting to note that this kind of discontinuous transition is not exclusive of the PG, but appears also in the p-spin-interaction spin-glass models [16] with $p>2$ and quadrupolar glass models [17], and also in the 'simplest spin glass' [18] and the random-energy model [19], suggesting some kind of universality.

(vi) Another interesting feature is that for the discontinuous situation ($p>4$), both the TAP approach and a dynamical study of the model [4] show the existence of another transition at a temperature $T_A > T_g$, signalled by a slowing down of the dynamical correlations as $T \to T_A^+$. Below T_A the system gets stuck in a metastable state. In the mean-field theory the barriers separating these metastable states are infinitely high, and a transition from ergodic to non-ergodic behaviour takes place. It will be shown below that the transition can be located in the static mean field theory as the point where the free energy is maximised when one of the variational mean-field parameters remains fixed at its physical endpoint. In this sense, a connection has been established [4, 16, 20] between the dynamical theories of the PG model and the p-spin-interaction model and theories of the structural glass transition. For example, it is usual to have in real glassy systems a temperature characterised by an important change in the transport properties (showing up in the slope of the Arrhenius plot of the viscosity), and a laboratory transition at a lower temperature in the metastable region, where the change in the heat capacity occurs, as in the PG case.

(vii) Because of the lack of equivalence between ferromagnetic and antiferromagnetic bonds with respect to their ability to constrain the states of coupled spins, the phase diagram is more complicated than the ISG [2]. In particular, for $p>4$ the system will freeze into a ferromagnetic state unless the average value of the random distribution of the bonds is negative (antiferromagnetic) and bigger in absolute value than some J_0. Throughout this work we will assume that this condition is always met.

(viii) The phase diagram exhibits more richness than the ISG and has, for example, mixed phases (longitudinal ferromagnetic ordering and transverse spin-glass behaviour) like the ones appearing in the vector spin glass [21], but most of its details are still not fully understood.

The relative simplicity of the mean-field solution for the spin-glass phase allows us to obtain explicitly the different types of fluctuating eigenmodes. Some of the corresponding eigenvalues, among them the ones which reduce in the $p=2$ case to the 'replicon' [22] modes which destabilise the Sherrington–Kirkpatrick (SK) solution [10, 12], although they are stable here, are slow modes whose fluctuations decay spatially with a correlation length $\xi_S \sim (T-T_g)^{-1}$. The other modes, which we call fast modes, are associated with correlation lengths $\xi_F \sim (T-T_g)^{-1/2}$.

In renormalisation group theory, the above failure of conventional scaling is due to a dangerous irrelevant variable. When the effect of the fluctuations is included, two different behaviours are found below eight dimensions (which is the upper critical dimension for this aspect of the model). For $p > p_0 \simeq 3.77$ the fluctuations are shown to restore the scaling behaviour at $d = 6$ by renormalising the slow fluctuations [23]. For $p < p_0$, and in particular for the physical case $p = 3$, we show that the fluctuations destabilise the solution (the trajectories in parameter space flow away from the region of stability); this probably implies a fluctuation-driven first-order transition [24].

These 'replicon' modes, even at the mean-field level, vanish at the 'dynamical' transition ($p > 4$), establishing a connection between the static and dynamic theories [5].

Using the knowledge of the eigenvalues of fluctuations, the corrections (to one-loop order) to the free energy and the equation of state can be calculated. Already between six and eight dimensions the effect of the remaining non-universal corrections also suggests the tendency to order by undergoing a first-order phase transition. This supports the picture that, as in the regular Potts models [25], in the Potts glass the critical p above which the transition changes from continuous to discontinuous is dimension dependent.

The plan of this paper is as follows. In § 2 we introduce the model and obtain the effective Hamiltonian. In § 3 we discuss the replica symmetric theory and its stability. In § 4 a replica symmetry breaking (RSB) solution is presented, and we discuss some of its features. In § 5 we consider the fluctuations around that solution, in the case of the continuous and the discontinuous transition.

2. The model

Our starting point is the simplex representation [26] of the Potts model in which in each site i of the lattice there is a spin $\boldsymbol{S}_i$ which can be in one of the p different states $\{\boldsymbol{e}^s\}$ $s = 1, 2, \ldots, p$. Each state is represented by a $(p-1)$-dimensional vector, and they point in the direction of the vertices of a hypertetrahedron in a $(p-1)$-dimensional vector space.

The vectors satisfy the following relations:

$$\sum_{s=1}^{p} e_a^s e_b^s = p\delta_{ab} \tag{2.1}$$

$$\sum_{a=1}^{p-1} e_a^s e_a^{s'} = p\delta_{ss'} - 1 \tag{2.2}$$

$$\sum_{s=1}^{p} e_a^s = 0. \tag{2.3}$$

The Hamiltonian of the Potts glass is

$$H = -\sum_{(ij)} J_{ij} \sum_{a=1}^{p-1} S_{i,a} S_{j,a} \tag{2.4}$$

where the sum is over all nearest-neighbour pairs (i, j). The bonds J_{ij} are randomly distributed, with a probability distribution

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}} \exp[-(J_{ij} - J_0)^2/2J^2]. \tag{2.5}$$

If we consider the quenched problem, where the disorder does not change in time (at least in the time scale of thermal equilibration), the averages over disorder configurations have to be taken on extensive quantities like the free energy, and not the partition function. This difficulty can be solved by using the replica method [14] in which the disorder-averaged partition function is evaluated for an n-fold replicated system, and then the free energy is obtained from its analytic continuation by

$$F=-kT\lim_{n\to 0}\frac{\overline{Z^n}-1}{n}$$

where overbars mean disorder averages.

Averaging over the distribution of the random bonds, the replicated partition function is

$$\overline{Z^n}=\operatorname*{Tr}_{\{S^\alpha\}}\exp\left[\frac{J^2}{2(kT)^2}\sum_{(ij)}\sum_{\alpha\neq\beta}S^\alpha_{i,a}S^\beta_{i,b}S^\alpha_{j,a}S^\beta_{j,b}+\frac{1}{kT}\left(J_0+\frac{p-2}{2}\frac{J^2}{kT}\right)\sum_{(ij)}\sum_{\alpha}S^\alpha_{i,a}S^\alpha_{j,a}\right] \tag{2.6}$$

where $S^\alpha_{i,a}$ stands for the a component of the α replica of the spin at site i, and use of the relation

$$(e^s\cdot e^{s'})^2=(p-2)e^s\cdot e^{s'}+p-1$$

was made, and irrelevant constants were neglected. The random variables J_{ij} have been eliminated at the price of coupling different replicas. A temperature-dependent enhancement in the ferromagnetic ordering field appears for $p>2$. The condition $J_0/J<1$ is necessary in the SK solution to ensure that the system freezes from the disordered phase into a spin-glass phase and not into conventional ferromagnetic order. The equivalent condition here would be

$$\frac{J_0}{J}+\frac{p-2}{2}\frac{J}{kT}<1.$$

At low enough temperature magnetic order will always be preferred. At the glass transition temperature ($kT_g=J$ in the infinite-range case) this condition becomes

$$\frac{J_0}{J}<\frac{4-p}{2}. \tag{2.7}$$

We see that for $p\geqslant 4$ an antiferromagnetic average of the bonds is necessary in order to freeze into a glassy phase; otherwise, it has been shown that the system orders ferromagnetically [2-3]. In this paper we will assume that the condition (2.7) is satisfied and we will focus on the spin-glass transition by setting the ferromagnetic order parameter to zero.

We disentangle the sum over spins at different sites by the usual Hubbard-Stratonovich transformation [21] introducing variables $Q^{\alpha\beta}_{ab}$ to obtain (neglecting multiplicative constants)

$$\overline{Z^n}=\int\prod_{\substack{(\alpha,\beta)\\ i,a,b}}(\mathrm{d}Q^{\alpha\beta}_{i,ab})\exp\left(-\frac{1}{2}\sum_{(\alpha,\beta)}(K^{-1})_{ij}Q^{\alpha\beta}_{i,ab}Q^{\alpha\beta}_{j,ab}\right)\operatorname*{Tr}_{\{S^\alpha_i\}}\exp\left(\sum_{\alpha,\beta}Q^{\alpha\beta}_{i,ab}S^\alpha_{i,a}S^\beta_{i,b}\right) \tag{2.8}$$

where $K_{ij}=(J/kT)^2$ for nearest neighbours and zero otherwise, and (α,β) means that each pair of different replicas should be included once.

We can now expand the second exponential keeping terms to fourth order (which are necessary to check the stability of the solution), evaluate the thermodynamic trace using the identities (2.1)-(2.3) and re-exponentiate the result to obtain:

$$\operatorname*{Tr}_{\{S^\alpha_i\}}\exp\left(\sum_{(\alpha,\beta)}Q^{\alpha\beta}_{i,ab}S^\alpha_{i,a}S^\beta_{i,b}\right)=\exp\left\{\sum_i\left[\frac{1}{4}\sum_{\alpha\neq\beta}(Q^{\alpha\beta}_{i,ab})^2+F_3\{Q^{\alpha\beta}_{i,ab}\}+F_4\{Q^{\alpha\beta}_{i,ab}\}\right]\right\} \tag{2.9a}$$

$$F_3\{Q^{\alpha\beta}_{i,ab}\}=\frac{1}{6}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}Q^{\alpha\beta}_{ab}Q^{\beta\gamma}_{bc}Q^{\gamma\alpha}_{ca}+\frac{1}{12}\sum_{\alpha\neq\beta}\frac{v_{abc}v_{def}}{p^2}Q^{\alpha\beta}_{ad}Q^{\alpha\beta}_{be}Q^{\alpha\beta}_{cf} \tag{2.9b}$$

$$F_4\{Q^{\alpha\beta}_{i,ab}\}=\frac{1}{8}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\left(\frac{F_{abcd}}{p}-\delta_{ab}\delta_{cd}\right)Q^{\alpha\beta}_{ae}Q^{\alpha\beta}_{be}Q^{\alpha\gamma}_{cf}Q^{\alpha\gamma}_{df}$$
$$+\frac{1}{4}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\frac{v_{abc}v_{def}}{p^2}Q^{\alpha\beta}_{ad}Q^{\alpha\beta}_{be}Q^{\alpha\gamma}_{cg}Q^{\beta\gamma}_{fg}+\frac{1}{8}\sum_{\substack{\alpha,\beta,\gamma,\delta\\ \text{all}\\ \text{different}}}Q^{\alpha\beta}_{ab}Q^{\beta\gamma}_{bc}Q^{\gamma\delta}_{cd}Q^{\delta\alpha}_{da}$$
$$+\frac{1}{48}\sum_{\alpha\neq\beta}\left(\frac{F_{abcd}F_{efgh}}{p^2}-3\delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh}\right)Q^{\alpha\beta}_{ae}Q^{\alpha\beta}_{bf}Q^{\alpha\beta}_{cg}Q^{\alpha\beta}_{dh}. \tag{2.9c}$$

We introduce the Potts tensors

$$v_{abc}=\sum_{s=1}^{p}e^s_a e^s_b e^s_c \tag{2.10a}$$

$$F_{abcd}=\sum_{s=1}^{p}e^s_a e^s_b e^s_c e^s_d. \tag{2.10b}$$

Some useful relations that these tensors satisfy are listed in appendix 1. We symmetrised the sums over replica variables by assuming

$$Q^{\alpha\beta}_{ab}=Q^{\beta\alpha}_{ba} \tag{2.11}$$

as suggested by the steepest descents result $\bar{Q}^{\alpha\beta}_{ab}\sim\langle S^\alpha_a S^\beta_b\rangle$ that can be obtained from (2.8).

The necessary steps for obtaining the effective Hamiltonian are a simple generalisation of the standard procedure for obtaining the continuum limit (see for example [22]). The first exponential in (2.8) is diagonal in the Fourier transform space, and

$$K(q)=\frac{1}{N}\sum_{i,j}\exp[\mathrm{i}\boldsymbol{q}(\boldsymbol{r}_i-\boldsymbol{r}_j)]K_{ij}=\left(\frac{J}{kT}\right)^2 z\left(1-\frac{1}{2z}\sum_{a}\boldsymbol{q}\cdot\boldsymbol{a}+\ldots\right)$$

when we are interested in the transition region dominated by the long-range fluctuations. (z is the coordination number and $\boldsymbol{a}$ are vectors joining one site of the lattice to its nearest neighbours.) Replacing the sums over sites ($\Sigma_i\rightarrow(1/a^d)\int \mathrm{d}^dx$) and using as unit of length the lattice spacing in a hypercubic lattice, we obtain the partition function as

$$Z^n=\int\left(\prod_{\substack{(\alpha,\beta)\\ a,b}}\mathrm{d}Q^{\alpha\beta}_{ab}(x)\right)\exp\left(-\int\mathrm{d}^dx\,H\{Q^{\alpha\beta}_{ab}(x)\}\right) \tag{2.12}$$

where the effective Hamiltonian functional is

$$H\{Q^{\alpha\beta}_{ab}(x)\}=\frac{1}{4}\left[\left(\frac{kT}{J\sqrt{z}}\right)^2-1\right]\sum_{\alpha\neq\beta}(Q^{\alpha\beta}_{ab}(x))^2$$
$$+\frac{1}{4}\left(\frac{kT}{Jz}\right)^2\sum_{\alpha\neq\beta}(\nabla Q^{\alpha\beta}_{ab}(x))^2-F_3\{Q^{\alpha\beta}_{ab}(x)\}-F_4\{Q^{\alpha\beta}_{ab}(x)\}.$$

A simple rescaling leads to the standard form

$$H\{q^{\alpha\beta}_{ab}(x)\}=-\tfrac{1}{4}t\sum_{\alpha\neq\beta}(Q^{\alpha\beta}_{ab}(x))^2+\frac{1}{4}\sum_{\alpha\neq\beta}(\nabla Q^{\alpha\beta}_{ab}(x))^2$$
$$-\frac{w_1}{6}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}Q^{\alpha\beta}_{ab}(x)Q^{\beta\gamma}_{bc}(x)Q^{\gamma\alpha}_{ca}(x)$$
$$-\frac{w_P}{12}\sum_{\alpha\neq\beta}\frac{v_{abc}v_{def}}{p^2}Q^{\alpha\beta}_{ad}(x)Q^{\alpha\beta}_{be}(x)Q^{\alpha\beta}_{cf}(x)$$
$$-yF_4\{Q^{\alpha\beta}_{ab}(x)\} \tag{2.13}$$

where $t = z - (kT/J)^2$ and the constants $w_{\rm I} = w_{\rm P}$ and y have the initial values $(Jz/kT)^3$ and $(Jz/kT)^4$ respectively. The subindices $w_{\rm I}$ and $w_{\rm P}$ indicate that the first cubic term also appears in an Ising spin glass [9], but the second one vanishes there due to the reflection symmetry of that model (see equation (3.1)). Its presence here is a feature of the Potts nature of the spins. At the mean-field level (and also when only fluctuations for $d > 6$ are considered) there is no need to distinguish between them, but below $d = 6$ they are renormalised in different ways [27, 28].

3. Replica-symmetric (RS) mean-field theory

As usual, we look for a stationary point of the effective Hamiltonian functional (2.13) with the order parameters constant in space. Furthermore, we impose the ansatz $Q_{ab}^{\alpha\beta}(x) = q^{\alpha\beta}\delta_{ab}$, i.e. we search for a homogeneous isotropic spin-glass phase. Inserting this in the free energy functional and performing the sum over Potts variables with the help of relations (A1.4), (A1.6) and (A1.8), we obtain

$$H_{\rm MF} = (p-1)\Bigg[-\frac{t}{4}\sum_{\alpha\neq\beta}(q^{\alpha\beta})^2 - \frac{w}{6}\sum_{\alpha\neq\beta\neq\gamma} q^{\alpha\beta}q^{\beta\gamma}q^{\gamma\alpha} - \frac{w(p-2)}{12}\sum_{\alpha\neq\beta}(q^{\alpha\beta})^3$$
$$-y\Bigg(\frac{p-2}{4}\sum_{\alpha\neq\beta\neq\gamma}(q^{\alpha\beta})^2 q^{\alpha\gamma}q^{\beta\gamma} + \frac{1}{8}\sum_{\substack{\alpha\beta\gamma\delta\\ \text{all}\\ \text{different}}} q^{\alpha\beta}q^{\beta\gamma}q^{\gamma\delta}q^{\delta\alpha}$$
$$+\tfrac{1}{48}(p^2-6p+6)\sum_{\alpha\neq\beta}(q^{\alpha\beta})^4\Bigg)\Bigg]. \tag{3.1}$$

We look for a replica-symmetric solution by assuming $q^{\alpha\beta} = q$ for all $\alpha \neq \beta$. We obtain

$$\frac{H_{\rm MF}}{n(n-1)} = (p-1)\Bigg[-\frac{t}{4}q^2 - \frac{w}{6}q^3\left(n-2+\frac{p-2}{2}\right)$$
$$-\frac{y}{48}q^4[12(p-2)(n-2)+6(n-2)(n-3)+(p^2-6p+6)]\Bigg] \tag{3.2}$$

$$\underset{(n\to 0)}{=} (p-1)\left(-\frac{t}{4}q^2 + \frac{6-p}{12}wq^3 - \frac{y}{48}q^4(p^2-30p+90) + {\rm O}(q^5)\right). \tag{3.3}$$

The saddle point equation associated with the expression (3.3) predicts a second-order phase transition at $t_c = 0$ for $p < 6$. Close to t_c ($t \ll 1$),

$$q = \frac{2t}{(6-p)w} + {\rm O}(t^2). \tag{3.4}$$

For $p > 6$, where the cubic term changes sign, a first-order transition is obtained. To perform the stability analysis of this solution we write

$$Q_{ab}^{\alpha\beta}(x) = q\delta_{ab} + R_{ab}^{\alpha\beta}(x)$$

and we expand (2.13) to second order in the fluctuations R. The $\frac{1}{2}n(n-1)(p-1)^2 \times \frac{1}{2}n(n-1)(p-1)^2$ stability matrix obtained using the relations (A1.3), (A1.5), (A1.7) and (A1.9) is

$$\tfrac{1}{8}(M_{abcd}^{\alpha\beta\gamma\nu} + M_{abdc}^{\alpha\beta\nu\gamma}) \tag{3.5}$$

$$M_{abcd}^{\alpha\beta\gamma\nu} = \delta_{ac}\delta_{bd}\left\{\left[-t+wq+yq^2\left(n-2+\frac{p-2}{2}\right)\right]\mathbf{A}_{\alpha\beta\gamma\nu} - \left[wq+yq^2\left(n-5+\frac{p-2}{2}\right)\right]\mathbf{B}_{\alpha\beta\gamma\nu} - yq^2\mathbf{C}_{\alpha\beta\gamma\nu}\right\}$$
$$-\frac{F_{abcd}}{p}\left\{\left[wq+yq^2\left((n-2)+\frac{p-2}{2}\right)\right]\mathbf{A}_{\alpha\beta\gamma\nu}+3yq^2\mathbf{B}_{\alpha\beta\gamma\nu}\right\}$$
$$+\delta_{ab}\delta_{cd}yq^2(\mathbf{A}_{\alpha\beta\gamma\nu}+\mathbf{B}_{\alpha\beta\gamma\nu}) \tag{3.6}$$

with the replica matrices given by

$$\mathbf{A}_{\alpha\beta\gamma\nu} = \delta_{\alpha\gamma}\delta_{\beta\nu} \tag{3.7a}$$

$$\mathbf{B}_{\alpha\beta\gamma\nu} = \delta_{\alpha\gamma}(1-\delta_{\beta\nu})+\delta_{\beta\nu}(1-\delta_{\alpha\gamma}) \tag{3.7b}$$

$$\mathbf{C}_{\alpha\beta\gamma\nu} = (1-\delta_{\alpha\gamma})(1-\delta_{\alpha\nu})(1-\delta_{\beta\gamma})(1-\delta_{\beta\nu}). \tag{3.7c}$$

The eigenvectors and eigenvalues (with their degeneracies) are explicitly obtained in appendix 2, as they will be useful also to discuss the stability of the replica-symmetry-breaking mean-field solution. Inserting the value of the order parameter close to the transition, (3.4) in the eigenvalues found there (A2.4a)-(A2.4c), we find

$$\lambda_{\mathrm{DR}} = t\frac{(2-p)}{6-p}+\mathrm{O}(t^2) \tag{3.8a}$$

$$\lambda_{\mathrm{VR}} = t\frac{(4-p)}{6-p}+\mathrm{O}(t^2). \tag{3.8b}$$

The Potts diagonal replicon mode is unstable at order t for any $p>2$. The Potts vector replicon mode is unstable only for $p>4$. All the other modes, not shown here, are stable for the whole range of validity of the solution $2<p<6$.

In the Ising case ($p=2$) the Potts indices disappear from the problem and only the Potts diagonal modes (DB, DA, DR) survive. We obtain there the three families of eigenvectors discussed by de Almeida and Thouless. In that case, the instability in the DR mode only appears at the order t^2, as can be seen in (3.8).

In the Potts case the instability coming from the replicon modes is stronger, as pointed out previously [2]. This should be contrasted with the lower level of replica symmetry breaking found in the stable solution in the next section.

4. Replica-symmetry-broken solution

Following Gross, Kanter and Sompolinsky [1], we try a replica-symmetry-breaking (RSB) ansatz for the mean-field $\{Q^{\alpha\beta}\}$ as follows. Group the n replicas in groups of m, where m is also a parameter (between n and 1) to be located by the saddle point equations. Then,

$$q^{\alpha\beta} = \begin{cases} q & \text{if } (\alpha,\beta) \text{ belong to the same group} \\ q_0 & \text{otherwise.} \end{cases} \tag{4.1}$$

This coincides with the first level in the Parisi RSB scheme [13], which is believed to be the correct solution to the long-range $p=2$ Sherrington–Kirkpatrick (SK) model. Just one level of RSB is known to give the correct solution for the p-spin interactions

($p \to \infty$) version of the SK problem [18]. It also locates a stable solution for the finite-p version of that model [4].

Using the ansatz (4.1) in (3.1) we obtain

$$\frac{H_{\mathrm{MF}}}{n(p-1)} = (1-m)\left[\frac{t}{4}q^2 + \frac{w}{6}q^3\left(m-2+\frac{p-2}{2}\right) + \frac{yq^4}{48}[12(p-2)(m-2)+6(m-2)(m-3)+(p^2-6p+6)]\right]. \tag{4.2}$$

In writing this expression we used the fact that the saddle point equation with respect to q_0 has the solution $q_0 = 0$ to all orders in q. This means that the replicas overlap with strength q, or they do not overlap at all. The saddle point equations are

$$q(1-m)\left[t + w\left(m-2+\frac{p-2}{2}\right)q + \frac{yq^2}{6}(p^2-30p+90+6m^2-54m+12mp)\right] = 0 \tag{4.3}$$

$$q^2\left[t + \frac{2}{3}w\left(2m-3+\frac{p-2}{2}\right)q + \frac{yq^2}{12}[p^2-42p+144+18m^2+24m(p-5)]\right] = 0. \tag{4.4}$$

For the case $p < 4$ they have the solution

$$q = \begin{cases} 0 & t<0 \quad (4.5a) \\ \dfrac{t}{w(4-p)} + \dfrac{yt^2}{w^3}\dfrac{\frac{31}{24}p^2 - 14p + \frac{59}{2}}{(4-p)^3} + \mathrm{O}(t^3) & t>0 \quad (4.5b) \end{cases}$$

$$m = \frac{p-2}{2} - \frac{yt}{w^2}\frac{(p^2+12p-36)}{8(4-p)} + \mathrm{O}(t^2).$$

For $p = 2$ we find $m = 0$ (no replica symmetry breaking) if we include in the original free energy functional terms up to q^3. It is only the presence of the quartic terms which gives the RSB [13]. The solution collapses at $p = 4$, when the cubic term in the free energy functional changes sign. Interestingly this coincides with $m(t=0) = 1$, which is the end point of the physical range of variation of the breaking point m.

The free energy of the spin-glass phase can be found by replacing (4.5b) in (4.2), and close to the transition it is

$$f = \frac{t^3}{24w(4-p)} + \mathrm{O}(t^4). \tag{4.6}$$

Below the transition the free energy is bigger than the free energy of the paramagnetic phase. This is a common feature of spin glasses. It is related here to the analytic continuation $n \to 0$, which involves a negative total number of replica pairs [10], and more generally to the nature of the SG transition as a blocking transition.

For $p > 4$ there is also a spin-glass solution to (4.3) and (4.4), and the transition can be located with the accessory condition $F_{\mathrm{PM}} = F_{\mathrm{SG}}$ which implies $m'_{\mathrm{g}} = 1$. The transition occurs at

$$t_{\mathrm{g}} = \frac{w^2}{3y}\frac{(p-4)^2}{(p^2-18p+42)} < 0 \tag{4.7a}$$

where the order parameter changes discontinuously from

$$q_g = \frac{2w}{y} \frac{(4-p)}{(p^2 - 18p + 42)} \tag{4.7b}$$

to zero. Although the transition is discontinuous it is not a conventional first-order transition. Close to the transition the free energy of the spin-glass phase is

$$f = q_g(t - t_g)^2 + O[(t - t_g)^3]. \tag{4.8}$$

Only its second derivative is finite at the transition, and there is no latent heat at t_g. Technically, as $F_{SG} > F_{PM}$ according to (4.8), any latent heat would have to be negative. Kirkpatrick and Wolynes [5] have argued that the physical reason for this unusual behaviour is that the effective number of relevant states into which the system can freeze becomes exponentially large when $T \to T_g^-$ because the complexity of the system diverges at that temperature. In this context it should be realised that, although in a Potts system it is easier to satisfy the requirements of conflicting bonds and therefore frustration generated by the configurational randomness must play a lesser role than in the Ising case, there is, on the other hand, an intrinsic disorder associated with the ground-state entropy of Potts antiferromagnets [29] which has to be taken into account. Later we will show that this transition, despite the thermodynamic properties discussed, does not have the characteristics of a critical point either.

Before discussing the stability of the solution found it should be pointed out that if we assume one more step of replica symmetry breaking (we divide each group of m replicas in m/m' subgroups of m' replicas each, and assume that replicas within a subgroup overlap as $Q^{\alpha\beta} = q'$) and we look for saddle point solutions for the new Hamiltonian with respect to q, q', m, m' some tedious algebra will show there is no new solution (beyond the trivial $q' = q$ or $m' = 1$) near $t = 0$. This absence of multistep RSB has been found also in the 'simplest spin glass' [18]. Furthermore, a solution along the lines of the Parisi work [13], where an infinite sequence of RSB levels is obtained and $(q, q', q'' \ldots) \to q(x)$ (the order parameter becomes a continuous function of the breaking point x), does not exist. A technical reason for this will be presented later.

5. Stability and fluctuations

To investigate the stability of the solution we write

$$Q_{ab}^{\alpha\beta} = q\delta_{ab}\delta_{G\alpha G\beta} + R_{ab}^{\alpha\beta}(x)$$

and expand (2.13) to second order in the fluctuations. The symbol $\delta_{G\alpha G\beta}$ is one if α and β belong to the same group and zero otherwise. The presence of this 'group Kronecker delta' greatly simplifies the stability matrix. The only non-vanishing terms $R_{ab}^{\alpha\beta} R_{cd}^{\gamma\nu}$ are:

(i) terms where α, β, γ, and ν all belong to the same group

(ii) terms where α and β belong to groups i and j, and γ and ν also belong to groups i and j.

The whole fluctuation matrix factorises in n/m identical submatrices $\mathbf{M}_1$, of dimension $\frac{1}{2}m(m-1)(p-1)^2 \times \frac{1}{2}m(m-1)(p-1)^2$ which couple intragroup fluctuations, and $\frac{1}{2}(n/m)[(n/m)-1]$ identical submatrices $\mathbf{M}_2$ (of dimension $m^2(p-1)^2 \times m^2(p-1)^2$) which couple intergroup fluctuations. The intragroup matrices $\mathbf{M}_2$ are exactly the same as the matrix (3.6) found in the replica-symmetric problem, with the difference that all the replica indices now run inside one group (between 1 and m) and m substitutes n everywhere in the expression (3.6). We can then use the solution to the eigenvalue problem found in appendix 2. The intergroup matrices $\mathbf{M}_2$ are

$$\mathbf{M}_{2abcd}^{\alpha\beta\gamma\nu} = -\delta_{ac}\delta_{bd}\left\{t\mathbf{A}^{\alpha\beta\gamma\nu} + \left[wq + yq^2\left(m-2+\frac{p-2}{2}\right)\right]\mathbf{B}^{\alpha\beta\gamma\nu} + yq^2\mathbf{C}^{\alpha\beta\gamma\nu}\right\}. \tag{5.1}$$

The solution to this eigenvalue problem is found in appendix 3.

5.1. Continuous transition

In the disordered phase ($q=0$) all eigenvalues are $-t$. The phase is stable in the region $T > T_c$ and unstable below the transition ($t>0$), as expected. The behaviour of the eigenvalues in the ordered phase is obtained replacing the values of the parameters (4.5b) in the expressions (A3.3)-(A3.16). Close to the transition the eigenvalues are

$$\lambda_{\mathrm{DB}} = t + \mathrm{O}(t^2) \tag{5.2a}$$

$$\lambda_{\mathrm{DA}} = \frac{t(6-p)}{2(4-p)} + \mathrm{O}(t^2) \tag{5.2b}$$

$$\lambda_{\mathrm{VB}} = \frac{t(5-p)}{4-p} + \mathrm{O}(t^2) \tag{5.2c}$$

$$\lambda_{\mathrm{VA}} = \frac{t(8-p)}{2(4-p)} + \mathrm{O}(t^2) \tag{5.2d}$$

$$\lambda_{\mathrm{VR}} = \frac{t}{4-p} + \mathrm{O}(t^2) \tag{5.2e}$$

$$\lambda_{\mathrm{TB}} = \frac{3t}{4-p} + \mathrm{O}(t^2) \tag{5.2f}$$

$$\lambda_{\mathrm{TA}} = \frac{t(p+4)}{2(4-p)} + \mathrm{O}(t^2) \tag{5.2g}$$

$$\lambda_{\mathrm{TR}} = \lambda_{\mathrm{AR'}} = \frac{t(p-1)}{4-p} + \mathrm{O}(t^2) \tag{5.2h}$$

$$\lambda_{\mathrm{AA'}} = \frac{tp}{2(4-p)} + \mathrm{O}(t^2) \tag{5.2i}$$

$$\lambda_{\mathrm{IA}} = \frac{t(p-2)}{2(4-p)} + \mathrm{O}(t^2) \tag{5.2j}$$

$$\lambda_{\mathrm{IR}} = \frac{t(p-2)}{4-p} + \mathrm{O}(t^2) \tag{5.2k}$$

$$\lambda_{\mathrm{DR}} = \lambda_{\mathrm{IB}} = \frac{yt^2}{24w^2(4-p)^2}(7p^2 - 24p + 12) + \mathrm{O}(t^3). \tag{5.2l}$$

The eigenvalues (5.2a)-(5.2k) are positive in the range of validity of the solution, $2 < p < 4$. The last two families of eigenvectors (DR and IB) have positive eigenvalues only for $p > p_0 \simeq 2.82$. It is in this range $p_0 < p < 4$ that we have, then, a stable mean-field theory with a continuous transition. This result was obtained first in [1]. Their statement that the quartic coefficient changes sign at p_0 in the theory with the Parisi simplification (where only the most dangerous term $-y(q^{\alpha\beta})^4$ is kept in the effective Hamiltonian

(2.13)), should be understood in the sense that in that case the soft eigenvalues (λ_{DR} and λ_{IB}) are equal to $-4yt^2/[w^2(4-p)^2]$ and a negative value has to be assigned to the constant y (for $p>p_0$) in order to obtain the correct stability properties. In this context it should be noted that it is this change in sign of the quartic coupling which plays the crucial role in ruling out a solution with an infinite number of RSB (*à la* Parisi), as can be checked inserting the full Parisi ansatz in the simplified effective Hamiltonian.

The presence of these soft modes indicates that the mean-field solution found violates scaling in the conventional sense. The propagators of these soft modes are $(k^2+\lambda_{soft})^{-1}$ and the corresponding correlations

$$\langle q_s(\boldsymbol{r})q_s(\boldsymbol{r}')\rangle \sim \exp[-(\boldsymbol{r}-\boldsymbol{r}')\sqrt{\lambda_s}]$$

have a correlation length that diverges at the transition as $\zeta_s \sim 1/t$, with an exponent $\nu=1$, while the other modes have the usual mean-field correlation length exponents $\nu=\frac{1}{2}$. It is interesting that one of these anomalous modes is the DR mode which, as shown before, is the one that destabilises the replica-symmetric solution, and in the $p=2$ case becomes the massless replicon mode [22]. It will be shown in a companion paper [30] that this soft mode proves to be a crucial one also for this solution. Indeed, as the eigenvalue λ_{DR} is proportional to the quartic coupling y, which is a dangerous irrelevant variable [23] it will be renormalised by fluctuations for $d<8$, and their effect will be to change the sign of the eigenvalue, rendering the solution unstable, for certain values of p. The other soft mode (IB), whose presence is intimately connected to the RSB ansatz adopted, is harmless for the continuous transition and it will be shown that the fluctuations below $d=8$ restore its scaling behaviour at the upper critical dimension $d=6$.

5.2. *Discontinuous transition*

As the transition is discontinuous for $p>4$, the perturbative approach is valid only when q_g is small. We can control the approximation by letting $p=4+\varepsilon$, $\varepsilon \ll 1$ (see equation (4.7*b*)). To leading order,

$$q_g = \frac{w}{y}\frac{\varepsilon}{7} \tag{5.3a}$$

$$t_g = -\frac{w^2}{y}\frac{\varepsilon^2}{42} \tag{5.3b}$$

in agreement with [5]. Using this value in the expressions (A3.3)–(A3.16) of the eigenvalues, it is found that all the fluctuations around the ordered phase are finite (and positive).

Interestingly, a similar feature to the one found in the continuous transition appears. All the fluctuation modes are proportional to ε to leading order, except for the soft modes (IB) and (DR) which are proportional to ε^2. There is a third mode (DB) which is also proportional to ε^2. This is a reflection of the accidental degeneracy between the breathing and replicon modes that occurs for $m=1$ as table 1 illustrates.

The finiteness of all the 'perturbative' fluctuations at T_g points to the fact that the discontinuous transition is not a conventional critical point, in spite of the continuity of the entropy and the unusual (for a first-order transition) thermodynamic properties, as there is no divergence of a correlation length and all the spin-glass susceptibilities remain finite.

Dynamical studies of the mean-field theory in the soft spin version of this model [4] showed the existence of another transition at a temperature $T_A > T_g$ whose signature is a critical slowing down of the kinetic correlations. A similar behaviour is found for the discontinuous transition ($p > 2$) in the p-spin interactions model. Kirkpatrick and Wolynes [5], through a Thouless–Anderson–Palmer approach, also found both transitions in the Potts ($p > 4$) case, and the dynamical transition appeared there when $q^{\alpha\beta} = 0$ for $\alpha \neq \beta$ and only the self-overlap $q^{\alpha\alpha}$ remained. This corresponds to the case of maximally broken replica symmetry. In the static approach presented here, this transition can be located with the assumption that the variational equation for m (equation (4.4)) is not satisfied and m is fixed at its physical end point $m(T = T_A) = 1$. Using this in (4.3), the variational equation for q, we obtain a solution which is first satisfied at

$$t_A = \frac{3}{8}\frac{w^2}{y}\frac{(p-4)^2}{(p^2-18p+42)} < t_g \qquad T_A > T_g. \tag{5.4a}$$

At that temperature,

$$q_A = \frac{3}{2}\frac{w}{y}\frac{(4-p)}{(p^2-18p+42)} < q_g. \tag{5.4b}$$

The derivation of this expression in the framework of the static perturbative approach is questionable since a factor $(1-m)$ was disregarded in the variational equation, despite the fact that we are interested in the $m = 1$ situation. This shortcoming is related to the fact that we are describing a state where no replica overlap remains, and the order parameter which describes the transition is a self-overlap which is not present in the formalism [4]. Nevertheless, the transition located this way is the same found in the dynamical approach and in the TAP approach.

We can also evaluate the fluctuation eigenvalue at this transition using (5.4) in the expressions (A3.3)-(A3.16).

All the eigenvalues exhibit the same behaviour found at the lower transition at T_g with the exception of the (DR) mode and its degenerate pair (DB) which vanish at T_A. Again the critical mode turns out to be the same as that which was a soft mode at the continuous transition and the replicon mode of the $p = 2$ problem. So the static approach actually exhibits the critical behaviour that was found in the dynamic approach. From the variational equation it follows that

$$q(T \leqslant T_A) = q_A + \left(\frac{[-6(t-t_A)]}{y(p^2-18p+42)}\right)^{1/2} \tag{5.5}$$

and close to the transition

$$\lambda_{DR} = \sqrt{-4t_A(t-t_A)} \tag{5.6}$$

from where a correlation length $\xi \sim (t-t_A)^{-1/4}$ is obtained.

This result is consistent with the picture introduced in [5], where it is argued that between T_A and the thermodynamic transition at T_g the system is frozen into a metastable state. At the mean-field level the barriers separating the metastable state are infinite and the system undergoes an ergodic to non-ergodic transition. Presumably in real glassy systems the barriers are finite; activated transitions between the different metastable states would explain the slowing down of the transport properties, and the hysteresis and other characteristics of glassy behaviour. It has been noted in this context that the presence of a temperature where there is a strong change of the transport properties while the thermodynamic properties change smoothly, is representative of many structural glasses [5].

In conclusion, we recapitulate the main results presented.

Starting from the Edwards-Anderson random bond Hamiltonian we obtained an effective Hamiltonian for the p-state Potts spin glass, keeping exact terms to order q^4. We found the RS solution and checked the stability matrix, showing that the solution is unstable to order q for any $p>2$. We introduced a simple RSB ansatz, in which the replicas overlap maximally or do not overlap at all, and found a continuous transition for $p<4$. We solved the stability matrix which factorises into intragroup and intergroup fluctuations and found that the solution is completely stable for $p>p_0$. While most modes are proportional to t, there are soft modes (proportional to t^2), and we interpreted this as a failure of scaling of the mean-field solution. For $p>4$ we found that there are two different transitions. The thermodynamic transition is discontinuous but there is no latent heat at T_g. All the perturbative fluctuations around it are finite. At a higher temperature T_A there is another transition found also in the dynamics, where one of the variational parameters is fixed at its physical endpoint. At that transition one of the soft modes becomes critical.

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Appendix 1

All the properties listed below are easily obtained using the definitions (2.10) and the relations (2.1)-(2.3) (summation convention is used):

$$v_{aab}=0 \tag{A1.1}$$

$$v_{abc}v_{ade}=pF_{bcde}-p^2\delta_{bc}\delta_{de} \tag{A1.2}$$

$$v_{abc}v_{abd}=p^2(p-2)\delta_{cd} \tag{A1.3}$$

$$v_{abc}^2=p^2(p-1)(p-2) \tag{A1.4}$$

$$F_{aabc}=p(p-1)\delta_{bc} \tag{A1.5}$$

$$F_{aabb}=p(p-1)^2 \tag{A1.6}$$

$$F_{abcd}F_{abef}=p^2\left(\frac{p-2}{p}F_{cdef}+\delta_{cd}\delta_{ef}\right) \tag{A1.7}$$

$$F_{abcd}^2=p^2(p-1)(p^2-3p+3) \tag{A1.8}$$

$$F_{abcd}v_{cde}=p(p-2)v_{abe}. \tag{A1.9}$$

Appendix 2

Because of the factorisation of each term of the matrix (3.6) it is natural to look for eigenvectors of the form $R_{cd}^{\gamma\nu}=R^{\gamma\nu}P_{cd}$, where the $R^{\gamma\nu}$ are $\frac{1}{2}n(n-1)$-dimensional simultaneous eigenvectors of the matrices **A**, **B** and **C** defined in expressions (3.7), and the

P_{cd} are $(p-1)^2$-dimensional eigenvectors of the Potts matrices F_{abcd} and $\delta_{ab}\delta_{cd}$ (and the diagonal $\delta_{ac}\delta_{bd}$).

As the Potts matrices are symmetric with respect to the exchange of the pair (c, d) of indices, the antisymmetric vectors

$$(P_A)_{cd} = \delta_{ca_0}\delta_{db_0} - \delta_{cb_0}\delta_{da_0} \qquad \underset{a_0 \neq b_0}{a_0, b_0 = 1, \ldots, p-1} \tag{A2.1a}$$

are eigenvectors with eigenvalue 0. We must look for the other eigenvectors in the orthogonal symmetric subspace $P_{cd} = P_{dc}$. Using relation (A1.5), the diagonal tensor $(P_D)_{cd} = \delta_{cd}$ is an eigenvector with eigenvalues $p(p-1)$ and $(p-1)$ respectively. Also, relations (A1.1) and (A1.9) show that the 'Potts vectors'

$$(P_V)_{cd} = v_{cde} \qquad e = 1, \ldots, p-1 \tag{A2.1b}$$

defined in (2.10) are eigenvectors with eigenvalues $p(p-2)$ and 0 respectively. The remaining family of eigenvectors can be obtained by imposing orthogonality conditions to the ones found previously. Each member of this family breaks the symmetry among Potts indices by selecting two components, and they will be called accordingly $(P_T)_{cd}$ ('Potts tensor' eigenvectors). Similarly, the $(P_V)_{cd}$ chooses one from among the Potts components. The expression of the $(P_T)_{cd}$ is rather cumbersome and will not be given here. We need only to know that their associated eigenvalues are both zero, and that, as matrices, they are symmetric and traceless. The degeneracies of the P_D, P_V, P_T and P_A eigenvalues are respectively 1, $p-1$, $\frac{1}{2}p(p-3)$ and $\frac{1}{2}(p-1)(p-2)$.

Inserting each solution for the Potts part in the general eigenvalue equation (3.5), we obtain eigenvalue equations for the replica vectors $R^{\alpha\beta}$. The equations obtained are linear combinations of

$$\tfrac{1}{2}(\mathbf{M}^{\alpha\beta\gamma\nu} \pm \mathbf{M}^{\alpha\beta\nu\gamma})$$

where the **M** matrices are the **A**, **B**, **C** replica matrices introduced in (3.7). The upper sign is obtained when we use the Potts symmetric eigenvectors (P_V, P_T, P_D) and the lower one when we use the antisymmetric ones (P_A), and thus we look for symmetric replica solutions in one case, and antisymmetric ones in the other, to maintain the overall symmetry given by (2.11). In the first case, the problem is exactly the same as that solved by de Almeida and Thouless [12], and the three families of replica vectors found are

$$\text{Breathing (B)} \quad \to R^{\alpha\beta} = R \text{ for all } (\alpha, \beta) \tag{A2.2a}$$

$$\text{Anomalous (A)} \quad \to R^{\alpha\beta} \begin{cases} \frac{1}{2}(2-n) & \text{if } \alpha \text{ or } \beta = \alpha_0 \qquad \alpha_0 = 1, \ldots, n \\ 1 & \text{otherwise} \end{cases} \tag{A2.2b}$$

$$\text{Replicon (R)} \quad \to R^{\alpha\beta} \begin{cases} \frac{1}{2}(n-2)(n-3) & \text{if } \alpha\beta = (\alpha_0, \beta_0) \\ \frac{1}{2}(3-n) & \text{if } \alpha = \alpha_0 \text{ and } \beta \neq \beta_0 \qquad \alpha_0 \neq \beta_0 = 1, \ldots, n \\ & \text{or vice versa} \\ 1 & \text{otherwise.} \end{cases} \tag{A2.2c}$$

For the antisymmetric case a slight modification of the de Almeida-Thouless case is found. There are only two families as homogeneous antisymmetric modes do not exist:

$$A' \to R^{\alpha\beta} = \delta_{\alpha\alpha_0} - \delta_{\alpha_0\beta} \qquad \alpha_0 = 1, \ldots, n \tag{A2.3a}$$

$$R' \to R^{\alpha\beta} = (\delta_{\alpha\alpha_0} - \delta_{\alpha_0\beta}) - (\delta_{\alpha\beta_0} - \delta_{\beta_0\beta}) - n(\delta_{\alpha\alpha_0}\delta_{\beta\beta_0} - \delta_{\alpha\beta_0}\delta_{\beta\alpha_0}) \qquad \alpha_0 \neq \beta_0 = 1, \ldots, n. \tag{A2.3b}$$

Table 1. Degeneracies and eigenvalues for each family of eigenvectors.

Family	Degeneracy	Eigenvalue of **A**	Eigenvalue of **B**	Eigenvalue of **C**
B	1	1	$2(n-2)$	$(n-2)(n-3)$
A	$n-1$	1	$n-4$	$-2(n-3)$
R	$\frac{1}{2}n(n-3)$	1	-2	2
A′	$n-1$	1	$n-2$	0
R′	$(n-1)(n-2)$	1	-2	0

In table 1 the degeneracy of each family and the eigenvalues corresponding to the three replica matrices **A**, **B**, **C** are listed.

We followed here the denomination introduced by Bray and Moore [21]. We obtain the different families of eigenvalues by combining the Potts and replica parts in all possible ways. (P_D, P_V, P_T) combines with (B, A, R) and leads to nine families (DB, DA, DR, …, TR), and P_A combined with (A′, R′) provides the last two (AA′, AR′). Using the eigenvalue information provided above and in table 1 together with the matrix (3.6) all the eigenvalues can be found. For example

$$\lambda_{DB} = -t - wq(p-2+2(n-2)) + O(q^2) \tag{A2.4a}$$

$$\lambda_{DR} = -t - wq(p-4) + O(q^2) \tag{A2.4b}$$

$$\lambda_{VR} = -t - wq(p-5) + O(q^2). \tag{A2.4c}$$

Appendix 3

The matrix (5.1) is diagonal in the Potts indices and its solutions are of the form

$$R^{\gamma\nu}_{cd} = \delta_{ca_0}\delta_{db_0}R^{\gamma\nu} \qquad a_0, b_0 = 1, \ldots, p-1. \tag{A3.1}$$

The solutions in replica space are similar to the ones found in (A2.2*a*)-(A2.2*c*), the difference coming from the fact that there is no symmetry between the two replica indices as they run over different sets of values (different groups). The three families of solutions are

$$\text{Breathing} \quad \rightarrow R^{\alpha\beta} = 1 \text{ for all } \alpha, \beta \tag{A3.2a}$$

$$\text{Anomalous} \quad \rightarrow R^{\alpha\beta}\begin{cases} 1-m & \text{if } \alpha = \alpha_0 \\ 1 & \text{otherwise} \end{cases} \qquad \alpha_0 = 1, \ldots, m \tag{A3.2b}$$

and

$$R^{\alpha\beta}\begin{cases} 1-m & \text{if } \beta = \beta_0 \\ 1 & \text{otherwise} \end{cases} \qquad \beta_0 = 1, \ldots, m$$

$$\text{Replicon} \quad \rightarrow R^{\alpha\beta}\begin{cases} (1-m)^2 & \text{if } \alpha = \alpha_0 \text{ and } \beta = \beta_0 \\ (1-m) & \text{if } \alpha = \alpha_0 \text{ and } \beta \neq \beta_0 \\ & \text{or vice versa} \\ 1 & \text{otherwise.} \end{cases} \qquad \alpha_0, \beta_0 = 1, \ldots, m \tag{A3.2c}$$

The degeneracies and the corresponding eigenvalues are given in table 2. In this way

Table 2. Degeneracies and eigenvalues for the intergroup coupling eigenvectors.

Family	Degeneracy	Eigenvalue of **A**	Eigenvalue of **B**	Eigenvalue of **C**
B	1	1	$2(m-1)$	$(m-1)^2$
A	$2(m-1)$	1	$m-2$	$1-m$
R	$(m-1)^2$	1	-2	1

we obtain three new intergroup families (IB, IA, IR). With all the information provided so far we can write the eigenvalues for the eleven intragroup families and the three intergroup families to order q^2. They are listed below to the order relevant to our work and the corresponding degeneracies are given:

$$\text{DB} \quad -t-wq[2(m-2)+p-2]+\mathrm{O}(q^2) \qquad \left(\frac{n}{m}\right) \tag{A3.3}$$

$$\text{DA} \quad -t-wq(m-4+p-2)+\mathrm{O}(q^2) \qquad \left(\frac{n(m-1)}{m}\right) \tag{A3.4}$$

$$\text{DR} \quad -t+wq(4-p)-yq^2[\tfrac{1}{2}p^2-10p+25+m(p-4)]+\mathrm{O}(q^3) \qquad \left(\frac{n(m-3)}{2}\right) \tag{A3.5}$$

$$\text{VB} \quad -t-wq[2(m-2)+p-3]+\mathrm{O}(q^2) \qquad \left(\frac{n}{m}(p-1)\right) \tag{A3.6}$$

$$\text{VA} \quad -t-wq(m-4+p-3)+\mathrm{O}(q^2) \qquad \left(\frac{n(m-1)}{m}(p-1)\right) \tag{A3.7}$$

$$\text{VR} \quad -t+wq(5-p)+\mathrm{O}(q^2) \qquad \left(\frac{n(m-3)}{2}(p-1)\right) \tag{A3.8}$$

$$\text{TB} \quad -t-wq(2m-5)+\mathrm{O}(q^2) \qquad \left(\frac{n}{m}\frac{p(p-3)}{2}\right) \tag{A3.9}$$

$$\text{TA} \quad -t-wq(m-5)+\mathrm{O}(q^2) \qquad \left(\frac{n}{m}(m-1)\frac{p(p-3)}{2}\right) \tag{A3.10}$$

$$\text{TR} \quad -t+3wq+\mathrm{O}(q^2) \qquad \left(\frac{n(m-3)}{2}\frac{p(p-3)}{2}\right) \tag{A3.11}$$

$$\text{AA}' \quad -t-wq(m-3)+\mathrm{O}(q^2) \qquad \left(\frac{n}{m}(m-1)\frac{(p-1)(p-2)}{2}\right) \tag{A3.12}$$

$$\text{AR}' \quad -t+3wq+\mathrm{O}(q^2) \qquad \left(\frac{n(m-1)(m-2)}{2}\frac{(p-1)(p-2)}{2}\right) \tag{A3.13}$$

$$\text{IB} \quad -t-wq2(m-1)-yq^2(m-1)(3m-5+p-2)+\mathrm{O}(q^3) \qquad \left(\frac{n}{2m}\left(\frac{n}{m}-1\right)(p-1)^2\right) \tag{A3.14}$$

$$\text{IA} \quad -t-wq(m-2)+\mathrm{O}(q^2) \qquad \left(\frac{n}{m}\left(\frac{n}{m}-1\right)(m-1)(p-1)^2\right) \tag{A3.15}$$

$$\text{IR} \quad -t+2wq+\mathrm{O}(q^2) \qquad \left(\frac{n}{2m}\left(\frac{n}{m}-1\right)(m-1)^2(p-1)^2\right). \tag{A3.16}$$

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